ON THE TRANSLATION INVARIANCE OF WAVELET SUBSPACES

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Abstract. An examination of the translation invariance of V_0 under dyadic rationals is presented, generating a new equivalence relation on the collection of wavelets. The equivalence classes under this relation are completely characterized in terms of the support of the Fourier transform of the wavelet. Using operator interpolation, it is shown that several equivalence classes are non-empty.

1. Introduction

A wavelet $\psi \in L^2(\mathbb{R})$ is a complete wandering vector for the unitary system $\{D^nT^l : n,\ l \in \mathbb{Z}\}$, i.e. the collection $\{D^nT^l\psi : n,\ l \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, where D, T are defined on $L^2(\mathbb{R})$ as: $Df(x) = \sqrt{2}f(2x)$ and $Tf(x) = f(x-1)$. Every wavelet can be associated with a *Generalized Multiresolution Analysis*, or GMRA (see [1]). Indeed, define the subspaces $V_j = \overline{span}\{D^nT^l\psi : n < j,\ l \in \mathbb{Z}\}$, then it is routine to verify that these subspaces satisfy the following four conditions:

1. $V_j \subset V_{j+1}$,
2. $DV_j = V_{j+1}$,
3. $\cap_{j\in\mathbb{Z}}V_j = \{0\}$ and $\cup_{j\in\mathbb{Z}}V_j$ has dense span in $L^2(\mathbb{R})$,
4. V_0 is invariant under T.

We shall call V_0 the *core space* for ψ. Item 4 is of interest since the core space is invariant under translations by the group $\mathbb{Z}$. A natural question is: are there other groups of translations under which V_0 is invariant? This paper will answer this question by looking at groups of translations by dyadic rationals.

Denote by T_α the unitary operator $T_\alpha f(x) = f(x-\alpha)$. T is to be understood as T_1. Note that $\widehat{T_\alpha} = M_{e^{-i\alpha\xi}}$. In this paper, we shall consider the groups of translations $\mathcal{G}_n = \{T_{\frac{m}{2^n}} : m \in \mathbb{Z}\}$, and the group $\mathcal{G}_\infty = \{T_\alpha : \alpha \in \mathbb{R}\}$. Denote by $\mathcal{L}_n$ the collection of all wavelets whose

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core space is invariant under $\mathcal{G}_n$. Note that these collections are nested:

$$\mathcal{L}_0 \supset \mathcal{L}_1 \supset \mathcal{L}_2 \ldots \supset \mathcal{L}_n \supset \mathcal{L}_{n+1} \supset \ldots \supset \mathcal{L}_\infty$$

We can then define an equivalence relation whose equivalence classes are given by $\mathcal{M}_n = \mathcal{L}_n - \mathcal{L}_{n+1}$, with $\mathcal{M}_\infty = \mathcal{L}_\infty$. Hence, $\mathcal{M}_n$ is the collection of all wavelets such that V_0 is invariant under $\mathcal{G}_n$ but not $\mathcal{G}_{n+1}$. The goal of this paper is to characterize these equivalence classes, while showing that several of them are not empty.

In general, V_0 can be quite complicated in structure. Indeed, it may not even be generated by translations of a finite number of functions. Hence, we wish to restrict our analysis to W_0. Recall that W_j is defined by $V_{j+1} = V_j \oplus W_j$. Clearly, $W_j = \overline{span}\{D^j T^l \psi : l \in \mathbb{Z}\}$.

Lemma 1. *Let $r < n$ be integers, and let $p = n - r$. The space V_r (resp. W_r) is invariant under $\mathcal{G}_k$ if and only if the space V_n (resp. W_n) is invariant under $\mathcal{G}_{k+p}$.*

Proof. By definition, $f \in V_r$ if and only if $D^p f \in V_n$. Suppose that $g \in V_n$ and define $f \in V_r$ such that $D^p f = g$. Consider the following commutation relation:

$$\begin{aligned} D^p T_{\frac{m}{2^k}} f(x) &= 2^{\frac{p}{2}} T_{\frac{m}{2^k}} f(2^p x) \\ &= 2^{\frac{p}{2}} f(2^p x + \frac{m}{2^k}) \\ &= D^p f(x + \frac{m}{2^{k+p}}) \\ &= T_{\frac{m}{2^{k+p}}} D^p f(x). \end{aligned}$$

This calculation establishes the statement. □

By lemma 1, another way to describe $\mathcal{M}_n$ is that $\psi \in \mathcal{M}_n$ if n is the largest integer such that V_{-n} is invariant under integral translations. If $\psi \in \mathcal{M}_n$, we shall say ψ has the *translation invariance of order n* property.

Theorem 2. *The core space V_0 for ψ is invariant under the action of $\mathcal{G}_n$ if and only if W_0 is invariant under the action of $\mathcal{G}_n$.*

If. Suppose that W_0 is invariant under $\mathcal{G}_n$. Then, by lemma 1, for $k > 0$, W_k is invariant under $\mathcal{G}_{n+k}$, whence $V_0^\perp = \oplus_{k=0}^\infty W_k$ is invariant under $\mathcal{G}_n$. If follows that V_0 is invariant under $\mathcal{G}_n$.

Only If. Suppose V_0 is invariant under $\mathcal{G}_n$. Then, again by lemma 1, V_1 is invariant under $\mathcal{G}_{n+1}$, and hence $\mathcal{G}_n$. Since $V_1 = V_0 \oplus W_0$, it follows that W_0 is also invariant under $\mathcal{G}_n$. □

For the purposes of this paper, we shall say that a set $E \subset \mathbb{R}$ is *partially self-similar* with respect to $\alpha \in \mathbb{R}$ if there exists a set F of non-zero measure such that both F and $F + \alpha$ are subsets of E. Additionally, if G, H are two subsets of $\mathbb{R}$, we shall say that G is 2π translation congruent to H if there exists a measurable partition G_n of G such that the collection $\{G_n + 2n\pi : n \in \mathbb{Z}\}$ forms a partition of H, modulo sets of measure zero. The letter λ will denote Lebesgue measure. Define a mapping $\tau : \mathbb{R} \to [0, 2\pi)$ such that $\tau(x) - x = 2\pi k$ for some integer k.

2. A Characterization of $\mathcal{M}_\infty$

Recall that a wavelet set is a set $W \subset \mathbb{R}$ such that the function ψ defined by $\hat{\psi} = \frac{1}{\sqrt{2\pi}}\chi_W$ is a wavelet. Such a wavelet ψ is called a Minimally Supported Frequency (MSF) wavelet. The following theorem reveals some structure of wavelet sets.

Theorem 3. *Let $f \in L^2(\mathbb{R})$, and let $E = supp(f)$. Then $\{e^{-inx} f(x) : n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(E)$ if and only if the following two conditions hold:*

1. *supp(f) is 2π translation congruent to $[0, 2\pi)$,*
2. *$|f(x)| = \frac{1}{\sqrt{2\pi}}$ for almost every x.*

This next theorem completely characterizes wavelet sets; the proof of which uses the previous theorem.

Theorem 4. *Let $W \subset \mathbb{R}$. Then W is a wavelet set if and only if the following two conditions hold:*

1. *W is 2π translation congruent to $[0, 2\pi)$,*
2. *$\cup_{j\in\mathbb{Z}} 2^j W = \mathbb{R}$*

modulo null sets.

The proof of both theorems can be found in [3], chapter 4.

Theorem 5. *Let ψ be a wavelet. Then, the following are equivalent:*

i) *ψ is a MSF wavelet,*
ii) *the subspace V_0 is invariant under translations by all real numbers,*
iii) *the subspaces V_j of the corresponding GMRA are invariant under integral translations.*
iv) *the subspaces W_j of the corresponding GMRA are invariant under integral translations.*

Proof. i) $\Rightarrow$ ii). If ψ is a MSF wavelet with wavelet set W, then by theorem 3, $\widehat{W_0} = L^2(W)$. Clearly, $\forall \alpha \in \mathbb{R}$, W_0 is invariant under

T_α since $\widehat{W_0}$ is invariant under multiplication by $e^{-i\alpha\cdot}$. It follows by theorem 2 that V_0 is invariant under all translations.

ii) $\Rightarrow$ iii). Since V_0 is invariant under $\mathcal{G}_n$ for all $n \geq 0$, by lemma 1, V_{-n} is invariant under $\mathcal{G}_0$.

iii) $\Rightarrow$ iv). By definition, $V_{j+1} = V_j \oplus W_j$. If both V_{j+1} and V_j are invariant under integral translations, it follows immediately that W_j is also invariant under integral translations.

iv) $\Rightarrow$ i). Let $\mathcal{C}$ be the collection of all operators for which W_0 is invariant. An easy calculation shows that $\mathcal{C}$ is WOT (weak operator topology) closed.

If W_j is invariant under integral translations, then again by lemma 1, W_0 is invariant under $\mathcal{G}_j$ for all j. Since $\cup_{n\geq 0}\mathcal{G}_n$ is dense in $\mathcal{G}_\infty$, in the strong operator topology, it follows that W_0 is invariant under $\mathcal{G}_\infty$. If we take the Fourier transform, then we get that $\widehat{W_0}$ is invariant under multiplication by $e^{-i\alpha\xi}$. The linear span of these operators are dense in the collection $\{M_h : h \in L^\infty(\mathbb{R})\}$ with respect to the WOT. It follows that $\widehat{W_0}$ is invariant under multiplication by any $L^\infty(\mathbb{R})$ function.

Next, we wish to show that $\widehat{W_0} = L^2(E)$, where $E = supp(\hat{\psi})$. First note that since $\{e^{-in\xi}\hat{\psi}(\xi)\}$ forms an orthonormal basis for $\widehat{W_0}$, $\hat{\psi}(\xi)$ has maximal support in the sense that if $\hat{f} \in \widehat{W_0}$, then the support of $\hat{f}$ is contained in the support of $\hat{\psi}$. This immediately implies that $\widehat{W_0} \subset L^2(E)$.

Let $g(\xi)$ be a compactly supported simple function, whose support F is contained in E. Define $E_n = \{\xi : \frac{1}{n-1} \geq \hat{\psi}(\xi) > \frac{1}{n}\}$, and define $F_n = F \cap E_n$. Since g is a simple function, it is uniformly bounded by some constant M. Let $\epsilon > 0$ be given. Choose an N such that $\lambda(\cup_{n>N}F_n) < \frac{\epsilon}{M}$, and define h_0 to be $\frac{g}{\hat{\psi}}\chi_{\cup_{n\leq N}F_n}$. Then, $h_0(\xi)\hat{\psi}(\xi) = g(\xi)$ on $\cup_{n\leq N}F_n$, so that $\|h_0\hat{\psi} - g\| < \epsilon$. Since $\widehat{W_0}$ is closed, $g \in \widehat{W_0}$; furthermore all such g's are dense in $L^2(E)$, whence $L^2(E) \subset \widehat{W_0}$.

Since $W_j \perp W_k$, $2^jE \cap E$ is a set of measure zero, and since $\oplus W_j$ is dense in $L^2(\mathbb{R})$, it follows that $\cup_j 2^jE = \mathbb{R}$. Furthermore, by theorem 3, E is 2π translation congruent to $[0, 2\pi)$, hence, by theorem 4, E is a wavelet set, and ψ is a MSF wavelet. □

Corollary 6. *The equivalence class $\mathcal{M}_\infty$ can be characterized in the following two ways:*

1. $\mathcal{M}_\infty = \cap_{n=0}^\infty \mathcal{L}_n$,
2. $\mathcal{M}_\infty$ *is precisely the collection of all MSF wavelets.*

Proof. By theorem 5, V_0 is invariant under $\mathcal{G}_n$ for all n if and only if V_0 is invariant under translations by all real numbers. This is equivalent to ψ being a MSF wavelet. $\square$

3. A Characterization of $\mathcal{M}_n$

Suppose ψ is a wavelet that is in $\mathcal{L}_1$. If $T_{\frac{m}{2}} f \in W_0$, then by taking the Fourier Transform, we have $e^{-i\frac{m}{2}\cdot}\hat{f} \in \widehat{W_0}$, and vice versa, so W_0 is invariant under translations by half integers if and only if $\widehat{W_0}$ is invariant under multiplication by $e^{-i\frac{m}{2}\cdot}$. Because of this, we shall proceed with the analysis in the frequency domain.

If $f \in W_0$, then we can write $f = \sum_{k\in\mathbb{Z}} c_k T^k \psi$, so taking the Fourier transform of both sides yields $\hat{f} = h\hat{\psi}$ for some $h \in L^2([0, 2\pi))$. Hence, we can describe $\widehat{W_0}$ by $\{h(\xi)\hat{\psi}(\xi) : h \in L^2([0, 2\pi))\}$. Suppose that $\xi \in E = supp(\hat{\psi})$. If $\widehat{W_0}$ is invariant under multiplication by $e^{-i\frac{m}{2}\xi}$, then for $m = 1$,

$$e^{-i\frac{1}{2}\xi} h(\xi)\hat{\psi}(\xi) = g(\xi)\hat{\psi}(\xi)$$

for some $g \in L^2([0, 2\pi))$. Note that if $\xi \in supp(\hat{\psi})$, then $e^{-i\frac{1}{2}\xi} h(\xi) = g(\xi)$. Let $\xi \in supp(\hat{\psi})$ and let k be an odd integer. Then,

$$\begin{aligned} g(\xi)\ \hat{\psi}(\xi + 2k\pi) &= g(\xi + 2k\pi)\ \hat{\psi}(\xi + 2k\pi) \\ &= e^{-i(\frac{1}{2})(\xi+2k\pi)}\ h(\xi + 2k\pi)\ \hat{\psi}(\xi + 2k\pi) \\ &= -e^{-i\frac{1}{2}\xi}\ h(\xi)\ \hat{\psi}(\xi + 2k\pi) \\ &= -g(\xi)\ \hat{\psi}(\xi + 2k\pi) \end{aligned}$$

This calculation shows that $\hat{\psi}$ cannot have both ξ and $\xi + 2k\pi$ in its support. We have established the first characterization theorem.

Theorem 7. *Let ψ be a wavelet. Then $\psi \in \mathcal{L}_1$ only if $E = supp(\hat{\psi})$ is not partially self similar with respect to any odd multiple of 2π.*

Corollary 8. *If $supp(\hat{\psi}) = \mathbb{R}$, then $\psi \in \mathcal{M}_0$.*

Corollary 9. *If ψ is compactly supported, then $\psi \in \mathcal{M}_0$.*

It is interesting to note that most of the wavelets used in practice have this property. It is unclear at this point if this has a meaningful interpretation from a numerical analysis point of view.

Theorem 7 extends to $\mathcal{L}_n$ in the following natural way.

Theorem 10. *Let ψ be a wavelet. Then $\psi \in \mathcal{L}_n$ only if the support of $\hat{\psi}$ is not partially self similar with respect to any odd multiple of $2^j\pi$ for all $j = 1, 2, \dots, n$.*

Proof. Let $\psi \in \mathcal{L}_n$. Hence,

$$e^{-i\frac{1}{2^n}\xi}h(\xi)\hat{\psi}(\xi) = g(\xi)\hat{\psi}(\xi)$$

for some $g \in L^2([0, 2\pi))$. Let $1 \leq j \leq n$, and let k be an odd integer. Then, by a similar computation,

$$\begin{aligned} g(\xi)\ \hat{\psi}(\xi + 2^jk\pi) &= g(\xi + 2^jk\pi)\ \hat{\psi}(\xi + 2k\pi) \\ &= e^{-i(\frac{1}{2^n})(\xi + 2^jk\pi)}\ h(\xi + 2^jk\pi)\ \hat{\psi}(\xi + 2^jk\pi) \\ &= e^{-i\frac{k}{2^{n-j}}\pi}e^{-i\frac{1}{2^n}\xi}\ h(\xi)\ \hat{\psi}(\xi + 2^jk\pi) \\ &= e^{-i\frac{k}{2^{n-j}}\pi}g(\xi)\ \hat{\psi}(\xi + 2^jk\pi) \end{aligned}$$

as above. □

We have now established necessary conditions for wavelets to be in the equivalence classes $\mathcal{M}_k$ for k not equal to 1 or ∞. This does not shed light onto whether such wavelets exist. Fortunately, to aid in the search, the converse of theorem 10 also holds.

Theorem 11. *Let ψ be a wavelet and let $E = supp(\hat{\psi})$ be such that it is not partially self similar with respect to any odd multiple of $2^j\pi$ for $j = 1, 2, \dots, n$. Then $\psi \in \mathcal{L}_n$.*

Proof. It suffices to show that

$$e^{-i\frac{1}{2^n}\xi}\hat{\psi}(\xi) = g(\xi)\hat{\psi}(\xi)$$

for some $g \in L^2([0, 2\pi))$.

Let $F \subset E$ be such that $\tau : F \to [0, 2\pi)$ is a bijection. (It can be easily shown that $\tau : E \to [0, 2\pi)$ is a surjection.) The injective property of τ can be assured in the following manner: for each $\xi \in [0, 2\pi)$, define the set $Z_\xi = \{m_\xi \in \mathbb{Z} : \xi + 2m_\xi\pi \in E\}$, then for ξ choose k_ξ to be 0 if $\xi \in E$, if not, choose $k_\xi = min\{m > 0 : minZ_\xi\}$, else choose $k_\xi = max\{m < 0 : m \in Z_\xi\}$. Let $F = \{\xi + 2k_\xi\pi : \xi \in [0, 2\pi)\}$. Note that by construction, F is 2π translation congruent to $[0, 2\pi)$. Hence,

$$e^{-i\frac{1}{2^n}\xi}\chi_F(\xi) = g(\xi)$$

where $g(\xi) \in L^2(F)$ and is 2π periodic. Thus, for $\xi \in F$,

$$e^{-i\frac{1}{2^n}\xi}\hat{\psi}(\xi) = g(\xi)\hat{\psi}(\xi).$$

For almost any $\xi \in E\backslash F$, there exists a $\xi' \in F$ and an integer l_ξ such that $\xi - \xi' = 2l_\xi\pi$. Moreover, by hypothesis, l_ξ is an even multiple of 2^n, since E is not partially self similar with respect to any odd multiple of $2^j\pi$. Since $e^{-i\frac{1}{2^n}\xi}$ is $2^n\pi$ periodic, we have that for $\xi \in E - F$,

$$\begin{aligned} e^{-i\frac{1}{2^n}\xi}\hat{\psi}(\xi) &= e^{-i\frac{1}{2^n}(\xi'+2l_\xi\pi)}\hat{\psi}(\xi' + 2l\xi\pi) \\ &= e^{-i\frac{1}{2^n}\xi'}\hat{\psi}(\xi' + 2l\xi\pi) \\ &= g(\xi')\hat{\psi}(\xi' + 2l\xi\pi) \\ &= g(\xi)\hat{\psi}(\xi). \end{aligned}$$

This completes the proof. □

We have established the following characterization of the $\mathcal{M}_n$'s.

Corollary 12. *The equivalence class $\mathcal{M}_n$ consists of all wavelets ψ such that the support of $\hat{\psi}$ is not partially self similar with respect to any odd multiples of $2^k\pi$, for $k = 1, 2, \dots, n$ but is partially self similar with respect to some odd multiple of $2^{n+1}\pi$.*

4. Examples

In this section, we will present examples of wavelets that are in the first four equivalence classes, with the last being in $\mathcal{M}_0$ but it is not an MRA wavelet, and hence cannot be compactly supported. The tool used to generate these wavelets is operator interpolation. Let ψ_{W_1} and ψ_{W_2} be MSF wavelets, with corresponding wavelet sets W_1 and W_2, respectively. By theorem 4, W_1 is 2π translation congruent to W_2. If $\sigma : W_1 \to W_2$ is effected by this translation congruence, then σ can be extended to a measurable bijection of $\mathbb{R}$ by defining $\sigma(x) = 2^{-n}\sigma(2^n x)$ where n is such that $2^n x \in W_1$.

If σ is *involutive*, i.e. σ^2 is the identity, and if h_1 and h_2 are measurable, essentially bounded, 2-dilation periodic functions (i.e. $h_1(2x) = h_1(x)$), then ψ defined by

$$\hat{\psi} = h_1\hat{\psi}_{W_1} + h_2\hat{\psi}_{W_2}$$

is again a wavelet provided the matrix

$$\begin{pmatrix} h_1 & h_2 \\ h_2 \circ \sigma^{-1} & h_1 \circ \sigma^{-1} \end{pmatrix} \tag{1}$$

is unitary almost everywhere. (Since σ^{-1} is 2-homogeneous, and the h_i's are 2-dilation periodic, in general it suffices to check this condition on W_1.) A complete discussion of this can be found in [3]. Note that the interpolated wavelet ψ has the property that *supp*$(\hat{\psi}) \subset W_1 \cup W_2$. Further, note that since σ on W_1 is given by translations by integral

multiples of 2π, σ completely describes the partial self similarity of $W_1 \cup W_2$ with respect to multiples of 2π.

In the following examples, σ will always be involutive.

Example 1. We shall now present an example of a wavelet in $\mathcal{M}_1$, which by corollary 12 is equivalent to $E = supp(\hat{\psi})$ being not partially self similar with respect to any odd multiples of 2π, but does have partially self similarity with respect to some multiple of 4π. Consider the following two wavelet sets:

$$\begin{aligned} W_1 &= [-\frac{8\pi}{7}, -\frac{4\pi}{7}) \cup [\frac{4\pi}{7}, \frac{6\pi}{7}) \cup [\frac{24\pi}{7}, \frac{32\pi}{7}) \\ W_2 &= [-\frac{8\pi}{7}, -\frac{4\pi}{7}) \cup [\frac{2\pi}{7}, \frac{3\pi}{7}) \cup [\frac{24\pi}{7}, \frac{30\pi}{7}) \\ &\quad \cup [\frac{31\pi}{7}, \frac{32\pi}{7}) \cup [\frac{60\pi}{7}, \frac{62\pi}{7}) \end{aligned}$$

A routine calculation shows:

$$\sigma(\xi) = \begin{cases} \xi, & \xi \in W_1 \cap W_2 \\ \xi - 4\pi, & \xi \in [\frac{30\pi}{7}, \frac{31\pi}{7}) \\ \xi + 8\pi, & \xi \in [\frac{4\pi}{7}, \frac{6\pi}{7}) \end{cases}$$

This σ is involutive. Indeed, since $\sigma([\frac{30\pi}{7}, \frac{31\pi}{7})) = [\frac{2\pi}{7}, \frac{3\pi}{7})$ and $[\frac{2\pi}{7}, \frac{3\pi}{7}) = 2[\frac{4\pi}{7}, \frac{6\pi}{7})$, for $\xi \in [\frac{30\pi}{7}, \frac{31\pi}{7})$, $\sigma^2(\xi) = \sigma(\xi - 4\pi) = \frac{1}{2}\sigma(2(\xi - 4\pi)) = \frac{1}{2}(2\xi - 8\pi + 8\pi) = \xi$. A similar computation shows that σ^2 is the identity on $[\frac{4\pi}{7}, \frac{6\pi}{7})$.

Construct h_1 and h_2 as follows:

$$\begin{aligned} h_1 &= \chi_{W_1 \cap W_2} + \frac{1}{\sqrt{2}} \chi_{[\frac{4\pi}{7}, \frac{6\pi}{7}) \cup [\frac{30\pi}{7}, \frac{31\pi}{7})} \\ h_2 &= \frac{1}{\sqrt{2}} \left(\chi_{[\frac{2\pi}{7}, \frac{3\pi}{7})} - \chi_{[\frac{60\pi}{7}, \frac{62\pi}{7})} \right) \end{aligned}$$

We need to check the condition of the matrix in equation 1.

It suffices to verify that the matrix is unitary on W_1. Clearly, on $W_1 \cap W_2$ the matrix is unitary, indeed it is the identity there. On $[\frac{30\pi}{7}, \frac{31\pi}{7})$, $h_1 = h_2 \circ \sigma^{-1} = \frac{1}{\sqrt{2}}$. Furthermore, if $\xi \in [\frac{30\pi}{7}, \frac{31\pi}{7})$, $h_2(\xi) = h_2(2\xi) = -\frac{1}{\sqrt{2}}$. Finally, $\sigma^{-1}(\xi) = \xi - 4\pi \in [\frac{2\pi}{7}, \frac{3\pi}{7}) = \frac{1}{2}[\frac{4\pi}{7}, \frac{6\pi}{7})$, hence $h_1 \circ \sigma^{-1}(\xi) = \frac{1}{\sqrt{2}}$. Thus, the matrix is simply:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

which is unitary as required. A similar computation shows that the matrix is also unitary on $[\frac{4\pi}{7}, \frac{6\pi}{7})$.

Example 2. Here we give an example of a wavelet in $\mathcal{M}_2$, which by corollary 12 is equivalent to $E = supp(\hat{\psi})$ being not partially self similar with respect to any odd multiples of 2π or 4π, but does have partially self similarity with respect to some multiple of 8π. Consider the following two wavelet sets:

$$\begin{aligned} W_1 &= [-8\pi, -\frac{112\pi}{15}) \cup [-\frac{16\pi}{15}, -\pi) \cup [-\frac{14\pi}{15}, -\frac{8\pi}{15}) \\ &\quad \cup [\frac{8\pi}{15}, \frac{14\pi}{15}) \cup [\pi, \frac{16\pi}{15}) \cup [\frac{112\pi}{15}, 8\pi) \\ W_2 &= [-8\pi, -\frac{112\pi}{15}) \cup [-\frac{14\pi}{15}, -\frac{\pi}{2}) \\ &\quad \cup [\frac{8\pi}{15}, \frac{14\pi}{15}) \cup [\frac{225\pi}{30}, 8\pi) \cup [\frac{224\pi}{15}, 15\pi) \end{aligned}$$

A routine calculation shows:

$$\sigma(\xi) = \begin{cases} \xi, & \xi \in W_1 \cap W_2 \\ \xi - 8\pi, & \xi \in [\frac{112\pi}{15}, \frac{225\pi}{30}) \\ \xi + 16\pi, & \xi \in [-\frac{16\pi}{15}, -\pi) \end{cases}$$

As in example 1, σ is involutive, and define h_1 and h_2 analogously:

$$\begin{aligned} h_1 &= \chi_{W_1 \cap W_2} + \frac{1}{\sqrt{2}} \chi_{[-\frac{16\pi}{15}, -\pi) \cup [\frac{112\pi}{15}, \frac{225\pi}{30})} \\ h_2 &= \frac{1}{\sqrt{2}} \left(\chi_{[-\frac{8\pi}{15}, -\frac{\pi}{2})} - \chi_{[\frac{224\pi}{15}, 15\pi)} \right) \end{aligned}$$

These functions satisfy 1.

Example 3. We shall now present an example of a wavelet in $\mathcal{M}_3$. Consider the following wavelet sets.

$$\begin{aligned} W_1 &= [-16\pi, -\frac{480\pi}{31}) \cup [-\frac{32\pi}{31}, -\pi) \cup [-\frac{30\pi}{31}, -\frac{16\pi}{31}) \\ &\quad \cup [\frac{16\pi}{31}, \frac{30\pi}{31}) \cup [\pi, \frac{32\pi}{31}) \cup [\frac{480\pi}{31}, 16\pi) \\ W_2 &= [-16\pi, -\frac{480\pi}{31}) \cup [-\frac{30\pi}{31}, -\frac{\pi}{2}) \\ &\quad \cup [\frac{16\pi}{31}, \frac{30\pi}{31}) \cup [\pi, \frac{32\pi}{31}) \cup [\frac{31\pi}{2}, 16\pi) \cup [\frac{960\pi}{31}, 31\pi) \end{aligned}$$

Then, σ is given by:

$$\sigma(\xi) = \begin{cases} \xi, & \xi \in W_1 \cap W_2 \\ \xi - 16\pi, & \xi \in [\frac{480\pi}{31}, \frac{31\pi}{2}) \\ \xi + 32\pi, & \xi \in [-\frac{32\pi}{31}, -\pi) \end{cases}$$

Again, as in example 1, σ is involutive; analogously define h_1 and h_2 as:

$$h_1 = \chi_{W_1 \cap W_2} + \frac{1}{\sqrt{2}} \chi_{[-\frac{32\pi}{31}, -\pi) \cup [\frac{480\pi}{31}, \frac{31\pi}{2})}$$

$$h_2 = \frac{1}{\sqrt{2}} \left(\chi_{[-\frac{16\pi}{31}, -\frac{\pi}{2})} - \chi_{[\frac{960\pi}{31}, 31\pi)} \right)$$

Example 4. In this example we shall construct a non-MRA wavelet in $\mathcal{M}_0$. Consider the following wavelet sets:

$$W_1 = [-\frac{32\pi}{7}, -\frac{28\pi}{7}) \cup [-\frac{7\pi}{7}, -\frac{4\pi}{7}) \cup [\frac{4\pi}{7}, \frac{7\pi}{7}) \cup [\frac{28\pi}{7}, \frac{32\pi}{7})$$

$$W_2 = [-\frac{8\pi}{7}, -\frac{4\pi}{7}) \cup [\frac{4\pi}{7}, \frac{6\pi}{7}) \cup [\frac{24\pi}{7}, \frac{32\pi}{7})$$

Both of these wavelets are non-MRA wavelets. It is shown in [8] that the interpolated wavelet also is not an MRA wavelet. We have that σ is given by:

$$\sigma(\xi) = \begin{cases} \xi, & \xi \in [-\frac{7\pi}{7}, -\frac{4\pi}{7}) \cup [\frac{4\pi}{7}, \frac{6\pi}{7}) \cup [\frac{28\pi}{7}, \frac{32\pi}{7}) \\ \xi - 2\pi, & \xi \in [\frac{6\pi}{7}, \frac{7\pi}{7}) \\ \xi + 8\pi, & \xi \in [-\frac{32\pi}{7}, -\frac{28\pi}{7}) \end{cases}$$

Construct h_1 and h_2 as follows:

$$h_1 = \chi_{W_1 \cap W_2} + \frac{1}{\sqrt{2}} \chi_{[-\frac{32\pi}{7}, -\frac{28\pi}{7}) \cup [\frac{6\pi}{7}, \frac{7\pi}{7})}$$

$$h_2 = \frac{1}{\sqrt{2}} \left(\chi_{[-\frac{8\pi}{7}, -\frac{7\pi}{7})} - \chi_{[\frac{24\pi}{7}, \frac{28\pi}{7})} \right)$$

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